MATH2040 Linear Algebra II

Tutorial 2

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1 Examples:

Example 1

Find the expression of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, which is the reflection of $\mathbb{R}^2$ about the line $y = 5x$.

Solution

Let α be the standard ordered basis of $\mathbb{R}^2$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \begin{pmatrix} -5 \\ 1 \end{pmatrix} \right\}$ to be another ordered basis of $\mathbb{R}^2$. Since T is the refection about the line $y = 5x$, so we have

$$T \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \quad \text{and} \quad T \begin{pmatrix} -5 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}.$$

Thus, $[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

By the change of coordinate formula,

$$[T]_\alpha = [I_V]_\beta^\alpha [T]_\beta [I_V]_\alpha^\beta.$$

Note, $[I_V]_\beta^\alpha = \begin{pmatrix} 1 & -5 \\ 5 & 1 \end{pmatrix}$ and $[I_V]_\alpha^\beta = \frac{1}{26} \begin{pmatrix} 1 & 5 \\ -5 & 1 \end{pmatrix}$

Therefore, the required expression is

$$T(x, y) = [T]_\alpha \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{26} \begin{pmatrix} 1 & 5 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{13} \begin{pmatrix} -12x + 5y \\ 5x + 12y \end{pmatrix}.$$

Example 2

Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 1 & -3 & 2 & -1 \\ -3 & 9 & -6 & 3 \\ 2 & -6 & 4 & -2 \\ -1 & 3 & -2 & 1 \end{pmatrix}$.

Solution The characteristic polynomial is $f(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -3 & 2 & -1 \\ -3 & 9-\lambda & -6 & 3 \\ 2 & -6 & 4-\lambda & -2 \\ -1 & 3 & -2 & 1-\lambda \end{vmatrix}$.

After doing elementary row and column operations,

$$\left|\begin{array}{cccc} 1-\lambda & -3 & 2 & -1 \\ -3 & 9-\lambda & -6 & 3 \\ 2 & -6 & 4-\lambda & -2 \\ -1 & 3 & -2 & 1-\lambda \end{array}\right| \xrightarrow[R_3 \to R_3 + 2R_4]{\substack{R_1 \to R_1 + R_4 \\ R_2 \to R_2 - 3R_4}} \left|\begin{array}{cccc} -\lambda & 0 & 0 & -\lambda \\ 0 & -\lambda & 0 & 3\lambda \\ 0 & 0 & -\lambda & -2\lambda \\ -1 & 3 & -2 & 1-\lambda \end{array}\right| \tag{1}$$

$$\xrightarrow[C_4 \to C_4 - 2C_3]{\substack{C_4 \to C_4 - C_1 \\ C_4 \to C_4 + 3C_2}} \left|\begin{array}{cccc} -\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ -1 & 3 & -2 & 15-\lambda \end{array}\right| \tag{2}$$

Set $f(\lambda) = 0$, we have $\lambda^3(15-\lambda) = 0$, so the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 15$.

To find the eigenvectors, we need to consider the eigenspaces of the two eigenvalues. And to simplify the computation, we could use the reduced form (1) of A.

$$\text{For } \lambda_1 = 0, E_{\lambda_1} = N(A - \lambda_1 I) = N \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 3 & -2 & 1 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -3 \end{pmatrix} \right\}.$$

$$\text{For } \lambda_2 = 15, E_{\lambda_2} = N(A - \lambda_2 I) = N \begin{pmatrix} -15 & 0 & 0 & -15 \\ 0 & -15 & 0 & 45 \\ 0 & 0 & -15 & -30 \\ -1 & 3 & -2 & -14 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} -1 \\ 3 \\ -2 \\ 1 \end{pmatrix} \right\}.$$

Remark: For this matrix A, if we let $Q = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & -2 \\ 1 & 2 & -3 & 1 \end{pmatrix}$, then $Q^{-1}AQ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 15 \end{pmatrix}$, which is a diagonal matrix.

2 Exercises:

Question 1 (Section 2.5 Q6(d)):

Let $A = \begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$. Let $L_A : \mathbb{F}^3 \to \mathbb{F}^3$ be a linear mapping defined by $L_A(x) = Ax$ for each column vector $x \in \mathbb{F}^3$. Then, compute $[L_A]_\beta$ and find an invertible matrix Q such that $[L_A]_\beta = Q^{-1}AQ$.

Question 2 (Section 2.5 Q7):

In $\mathbb{R}^2$, let L be the line $y = mx$, where $m \neq 0$. Find an expression for $T(x, y)$, where

(a) T is the reflection of $\mathbb{R}^2$ about L.

(b) T is the projection on L along the line perpendicular to L.

Question 3 (Section 5.1 Q2(f)):

Let $V = M_{2\times 2}(\mathbb{R})$ and $T : V \to V$ be defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -7a-4b+4c-4d & b \\ -8a-4b+5c-4d & d \end{pmatrix}$. If an ordered basis $\beta = \left\{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}\right\}$, compute $[T]_\beta$. Also, determine whether β is a basis consisting of eigenvectors of T.

Solution

(Please refer to the practice problem set 2.)